Freshman Meet 3 – April 3, 2013
Round 1: Graphing on a Number Line

NO CALCULATOR ALLOWED

Draw the graph of each of the following inequalities on the corresponding number line provided below. Please specify all endpoints on your graph. Your graph need <u>*not*</u> *be drawn to scale.*

1. $-2x + 4 \leq 3x - 1$

2. $-2|x + 3| \leq -10$

3. $\dfrac{x+2}{x-8} < \dfrac{x+9}{x-3}$

<u>ANSWERS</u>

(1 pt.) 1.

(2 pts.) 2.

(3 pts.) 3.

Freshman Meet 3 – April 3, 2013
Round 2: Operations on Polynomials

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. Solve for all values of x:

$$x(x+8) - x(x+3) - 23 = 3x + 11$$

2. Simplify to a single polynomial:

$$(x^4 + x^3 + x^2 + x + 1)(x^2 - 1) \div (x + 1)$$

3. Let k be a positive number. If you expand $(3x-k)^3$ and add the coefficients of x^2 and x, you get a sum of 90. Find the value of k.

ANSWERS

(1 pt.) 1. ____________________

(2 pts.) 2. ____________________

(3 pts.) 3. ____________________

Freshman Meet 3 – April 3, 2013
Round 3: Techniques of Counting and Probability

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. There were 20 competitors at a math contest. Each competitor shook hands once with every other competitor. How many handshakes took place?

2. Suppose you toss a fair coin 7 times. What is the probability that you will get at least 4 heads?

3. A bag contains 2 red and 3 blue marbles. A marble is drawn and not replaced. A second marble is then drawn. Find the probability that the marbles are different colors.

ANSWERS

(1 pt.) 1. ____________________

(2 pts.) 2. ____________________

(3 pts.) 3. ____________________

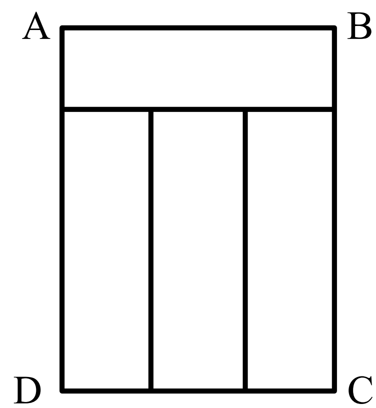
Freshman Meet 3 – April 3, 2013
Round 4: Perimeter, Area, and Volume

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. The base of a rectangular solid measures 7 by 12 units. Find the altitude (height) if the volume is 504 cubic units.

2. Four congruent rectangles are placed as shown to form a new rectangle $ABCD$ with area 48. What is the perimeter of $ABCD$?



3. A right rectangular prism is 10 feet long and 4 feet wide. The number of square feet in its surface area is numerically equal to the number of cubic feet in its volume. In feet, what is the height of the prism?

ANSWERS

(1 pt.) 1. ________________

(2 pts.) 2. ________________

(3 pts.) 3. ________________ feet

Freshman Meet 3 – April 3, 2013
TEAM ROUND

All answers must either be in simplest exact form or rounded to EXACTLY three decimal places, unless stated otherwise. (3 POINTS EACH)

APPROVED CALCULATORS ALLOWED

1. The area of a 120-degree sector of a circle is 27π. Find the circumference of the circle.
2. When the length of a square is increased by 7 inches and its width is cut in half, its area is decreased by 4 square inches. Find the area (in square inches) of of the newly formed rectangle.
3. The number 132 is divisible by how many positive integers? (Include 1 and itself.)
4. The sum of two numbers is 24 and the sum of their reciprocals is 8/45. Find the product of the two numbers.
5. Factor over the real numbers:
$$4x^4 + 1$$
6. What is the remainder when 7^{219} is divided by 100?
7. A six-story building has an underground parking garage. Three people are in the garage waiting for the elevator. Assume that each person is equally likely to get off at any floor. What is the probability that they all get out on different floors? Express your answer as a fraction.
8. *Draw the graph of the following inequality of the number line provided on the answer sheet. Please specify all endpoints on your graph. Your graph need* <u>*not*</u> *be drawn to scale.*
$$|5 - 2x| > 3x - 10$$

Freshman Meet 3 – April 3, 2013
TEAM ROUND ANSWER SHEET

1. ____________________

2. ____________________ in^2

3. ____________________

4. ____________________

5. ____________________

6. ____________________

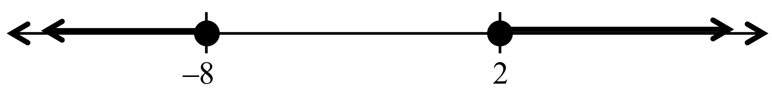
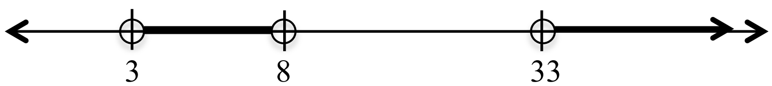
7. ____________________

8.

WORCESTER COUNTY MATHEMATICS LEAGUE

Freshman Meet 3 – April 3, 2013
ANSWERS

ROUND 1
(Bartlett, South, Auburn)

1. 
2. 
3. 

ROUND 2
(Shepherd Hill, St. John's, Worc Acad)

1. $x = 17$
2. $x^5 - 1$
3. $k = 5$

ROUND 3
(Algonquin, Westborough, St. Peter-Marian)

1. 190
2. $1/2 = 0.5 = 50\%$
3. $3/5 = 0.6 = 60\%$

ROUND 4
(Bartlett, Southbridge, Worc Acad)

1. 6
2. 28
3. $20/3 = 6\frac{2}{3} = 6.\overline{6}$

TEAM ROUND
(Shrewsbury, Algonquin, St. John's, Shepherd Hill, Burncoat, Bromfield, Hopedale, St. John's)

1. $18\pi \approx 56.549$
2. 60 in^2
3. 12
4. 135
5. $(2x^2 - 2x + 1)(2x^2 + 2x + 1)$
6. 43
7. $5/9$
8. 

Freshman Meet 3 – April 3, 2013
FULL SOLUTIONS

ROUND 1

1. We have

$$\begin{aligned} -2x+4 &\le 3x-1 \\ 5 &\le 5x \\ 1 &\le x. \end{aligned}$$

2. Remember to flip the inequality when multiplying (or dividing) by a negative number.

$$\begin{aligned} -2|x+3| &\le -10 \\ |x+3| &\ge 5. \end{aligned}$$

First case: $x+3 \ge 5 \implies x \ge 2$.

Second case: $x+3 \le -5 \implies x \le -8$.

3. We find the critical points of the inequality. The first critical point is

$$\begin{aligned} \frac{x+2}{x-8} &= \frac{x+9}{x-3} \\ (x+2)(x-3) &= (x-8)(x+9) \\ x^2-x-6 &= x^2+x-72 \\ 66 &= 2x \\ 33 &= x. \end{aligned}$$

However, since there are fractions, other critical points are when either fraction is undefined. Here, those points are at 8 and 3.

We now must test points in each region. Convenient choices are 0, 5, 10, and 38.

$x=0$:

$$\begin{aligned} \frac{2}{-8} &\ ?\ \frac{9}{-3} \\ -\frac{1}{4} &> -\frac{1}{3} \end{aligned}$$

$x = 5$:

$$\frac{7}{-3} < \frac{14}{2}$$

$x = 10$:

$$\frac{12}{2} > \frac{19}{7}$$

$x = 38$:

$$\begin{array}{ccc} \frac{40}{30} & ? & \frac{47}{35} \\ \frac{4}{3} & ? & \frac{47}{35} \\ 140 & < & 141 \end{array}$$

Therefore, our answer is $3 < x < 8$ or $x > 33$.

ROUND 2

1. Multiply out:

$$\begin{aligned} x^2 + 8x - x^2 - 3x - 23 &= 3x + 11 \\ 5x - 23 &= 3x + 11 \\ 2x &= 34 \\ x &= \boxed{17} \end{aligned}$$

A shortcut is to subtract $(x + 3)$ from $(x + 8)$ before multiplying by x.

2. We recognize that $(x^2 - 1) \div (x + 1) = (x - 1)$ and recall the factorization

$$(x^n - 1) = (x - 1)(x^{n-1} + x^{n-2} + \ldots + x + 1).$$

Our problem has $n = 5$, so the answer is $\boxed{x^5 - 1}$.

3. Expanding, $(3x - k)^3 = 27x^3 - 27kx^2 + 9k^2x - k^3$. The sum of the coefficients of the x^2 and x terms is $-27k + 9k^2$. This is given to be 90, so

$$\begin{aligned} -27k + 9k^2 &= 90 \\ k^2 - 3k - 10 &= 0 \\ (k - 5)(k + 2) &= 0 \end{aligned}$$

Since we are given that k is positive, it is equal to $\boxed{5}$.

ROUND 3

1. **METHOD I:** Each of the 20 competitors shakes hands with the 19 others, for a total of $20 \cdot 19 = 380$. However, this counts each handshake twice (once from each "side" of the handshake). Therefore, the answer is $380/2 = \boxed{190}$.

   METHOD II: A handshake happens between each distinct pair of competitors. The number of pairs is ${}_{20}C_2 = \dfrac{20 \cdot 19}{2} = \boxed{190}$.

2. Recognize the symmetry of the problem. After 7 coin tosses, you either get at least 4 heads or you get at most 3 heads. The latter is equivalent to getting at least 4 tails. Since this is a fair coin, the probability of getting at least 4 heads is equal to the probability of getting at least 4 tails. Therefore, each has probability $\boxed{1/2}$.

3. The total number of ways to draw two marbles is $5 \cdot 4 = 20$. The complement of the probability we want is the chance that both marbles are the same color. The number of ways for that is $2 \cdot 1$ (both red)$+3 \cdot 2$ (both blue) $= 8$. Therefore, the desired probability is $1 - 8/20 = \boxed{3/5}$.

ROUND 4

1. The arithmetic is easier if you divide one dimension into the volume before multiplying out the area of the base. The altitude is $504 \div 7 \div 12 = 72 \div 12 = \boxed{6}$.

2. By the arrangement in the figure, the length and width of the smaller rectangles are in a $3:1$ ratio. Let the side lengths of the small rectangles be $3x$ and x. Then, since $ABCD$ has area 48, each small rectangle must have area $48/4 = 12$. Therefore, $(3x)(x) = 12$ and $x = 2$. Thus $ABCD$ has side lengths 8 and 6 so its perimeter is $2(8+6) = \boxed{28}$.

3. Let the height of the prism be x. Then, the surface area is $2(40 + 10x + 4x)$ and the volume is $40x$. Equating these,

$$\begin{aligned} 28x + 80 &= 40x \\ 80 &= 12x \\ \boxed{20/3} &= x. \end{aligned}$$

TEAM ROUND

1. A 120-degree sector is one-third of the circle, so the area of the circle is 81π. Since $A = \pi r^2$, the radius is 9 and therefore the perimeter is $2\pi r = \boxed{18\pi}$.

2. Let the length of the original square be x inches. Then,

$$\begin{aligned}(x+7)(x/2)+4 &= x^2\\ \frac{1}{2}x^2+\frac{7}{2}x+\frac{8}{2} &= x^2\\ x^2-7x-8 &= 0\\ (x-8)(x+1) &= 0\end{aligned}$$

Since this is a geometry problem and x is a length, it must be positive, so $x = 8$. Therefore, the area of the newly formed rectangle is $\boxed{60}$.

3. The prime factorization of 132 is $132 = 2^2 \cdot 3^1 \cdot 11^1$. A divisor of 132 will be some combination of these prime factors: we can choose to have 0, 1, or 2 factors of 2; 0 or 1 factors of 3; and 0 or 1 factors of 11. This gives a total of $(2+1)(1+1)(1+1) = \boxed{12}$ divisors.

4. **METHOD I:** Let x and y be the two numbers. Then, $x + y = 24$ and $\frac{1}{x}+\frac{1}{y}=\frac{8}{45}$. Substitute $y = 24 - x$ to find

$$\begin{aligned}\frac{1}{x}+\frac{1}{24-x} &= \frac{8}{45}\\ \frac{24-x+x}{x(24-x)} &= \frac{8}{45}\\ 24\cdot 45 &= 8x(24-x)\\ 3\cdot 45 &= x(24-x)\\ \boxed{135} &= xy\end{aligned}$$

(Continuing to solve would yield $x = 9, 15$).

METHOD II: Let x and y be the two numbers. We are given that $24 = x + y$ and $\frac{8}{45}=\frac{1}{x}+\frac{1}{y}=\frac{x+y}{xy}$. Plugging in,

$$\begin{aligned}\frac{8}{45} &= \frac{24}{xy}\\ xy &= \frac{24\cdot 45}{8}\\ xy &= \boxed{135}.\end{aligned}$$

5. Notice that $4x^4+1$ is almost a perfect square. Let us creatively add zero to complete the square and also set up a difference of squares situation:

$$\begin{aligned} 4x^4+1 &= 4x^4+4x^2+1-4x^2 \\ &= (2x^2+1)^2-(2x)^2 \\ &= \boxed{(2x^2-2x+1)(2x^2+2x+1)} \end{aligned}$$

6. **METHOD I:** Using a calculator, explore the last two digits of powers of 7:

n	last two digits of 7^n
0	01
1	07
2	49
3	43
4	01
5	07
6	49
7	43
8	01

Notice the pattern that repeats every 4 powers of 7. Following the pattern, the last two digits of $7^{216}=7^{4\times 54}$ are 01, so the last two digits of 7^{219} are the same as the last two digits of 7^3, or $\boxed{43}$.

METHOD II: This can be made more formal using modular arithmetic. Since $7^4 \equiv 1 \pmod{100}$, we know that $7^{4n} \equiv 1 \pmod{100}$ for any positive integer n. Hence $7^{219} \equiv 7^3 \equiv 43 \pmod{100}$.

7. There are a total of $6^3=216$ possibilities. The number of ways for the three people to get off at different floors is $6\cdot 5\cdot 4=120$. Therefore, the probability is $120/216=\boxed{5/9}$.

8. There are two possibilities we must consider with the absolute value.

Case 1:

$$\begin{aligned} 5-2x &> 3x-10 \\ 15 &> 5x \\ 3 &> x \end{aligned}$$

Case 2:

$$\begin{aligned} 5-2x &< -3x+10 \\ x &< 5 \end{aligned}$$

Since absolute value inequalities lead to OR combinations and not AND combinations, the answer is $x<5$.